Solutions to Question Sheet 7, Differentiation II. v1 2019-20

Mean Value Results

We look at applications of Rolle's Theorem, the Mean Value Theorem and Cauchy's Mean Value Theorem.

1. State Rolle's Theorem.

   Suppose that $f(x)$ is continuous on $[a,b]$, is twice differentiable on (a,b) and has three distinct zeros in $[a,b]$. Prove that there exists $c \in (a,b)$ such that $f''(c) = 0$.

   Can you generalise this result? Can you give a proof of your general result?

   (**Hint**: For a general result use proof by induction)

   Solution ***Rolle's Theorem*** *If f is differentiable on the open interval (a,b), continuous on the closed interval $[a,b]$ and $f(a) = f(b)$ then there exists a $c : a < c < b$ such that $f'(c) = 0$.*

   Let the three zeros of f be $a_1 < a_2 < a_3$.

   Apply Rolle's Theorem to f on $[a_1, a_2]$ to find $a_1 < c_1 < a_2$ for which $f'(c_1) = 0$.

   Apply Rolle's Theorem to f on $[a_2, a_3]$ to find $a_2 < c_2 < a_3$ for which $f'(c_2) = 0$.

   Note that $c_1 < a_2 < c_2$ so $c_1 \neq c_2$.

   Finally, apply Rolle's Theorem to f' on $[c_1, c_2]$ to find $c_1 < c < c_2$ for which $(f')'(c) = 0$ i.e. $f''(c) = 0$.

   The generalised result would be: *Suppose that f is continuous on $[a,b]$, with n derivatives on (a,b) and has $n+1$ distinct zeros in $[a,b]$. Then there exists $c \in (a,b)$ such that $f^{(n)}(c) = 0$.*

   Proof The base case, $n = 1$, is Rolle's Theorem. Assume the statement is true when $n = k \geq 1$, so the result holds for all functions continuous on $[a,b]$, that have k derivatives and $k+1$ zeros in $[a,b]$.

Let f be a function continuous on $[a, b]$, with $k+1$ derivatives and $k+2$ zeros in $[a, b]$.

Rolle's Theorem implies that f' has a zero between each consecutive pairs of zeros of f, hence f' has $k+1$ zeros. Further f' is continuous because it is differentiable and it has, in fact, k derivatives. Thus we can apply the inductive hypothesis to f' to deduce the existence of $c \in (a, b)$ such that $(f')^{(k)}(c) = 0$, that is, $f^{(k+1)}(c) = 0$.

Hence the statement is true for $n = k+1$. Therefore, by induction, the result holds for all $n \geq 1$.

2. In Question 10 on Sheet 4 it was seen that $e^x = 4x^2$ has at least 3 real solutions. Prove that it has **exactly** 3 real distinct solutions.

   Solution Assume that $e^x = 4x^2$ has **4 or more** real solutions.

   Let $f(x) = e^x - 4x^2$, so we are assuming that f has *at least* 4 real distinct zeros. Take 4 of its zeros.

   By the ideas of the previous question, f' has at least 3 zeros, there being one between each pair of consecutive zeros of f. Then, repeating, f'' has at least 2 zeros and f''' has at least 1 zero. But $f'''(x) = e^x$ which has no real zero. Therefore the assumption is false, and so $f(x)$ has at *most* 3 real solutions. Yet from Sheet 6 we know that f has at *least* 3 zeros, therefore f has exactly 3 real solutions.

3. State the Mean Value Theorem.

   In each of the following examples find, if possible, an explicit value for $c \in (a, b)$ satisfying the *formula* of the Mean Value Theorem.

   In each case explain whether the Mean Value Theorem can be applied or not.

   i) $f(x) = x(x-2)(x-4)$, on $[1, 3]$.

   ii) $f(x) = 1/x^2$, on $[-1, 1]$.

   iii) $f(x) = x^{1/3}$, on $[-1, 1]$.

Solution ***Mean Value Theorem*** *If the function f is differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$ then there exists $c : a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

i) The formula (1) with $f(x) = x\,(x-2)\,(x-4)$ on $[1,3]$ becomes

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{-3 - 3}{2} = -3.$$

But

$$\begin{aligned} f'(x) &= (x-2)\,(x-4) + x\,(x-4) + x\,(x-2) \\ &= 3x^2 - 12x + 8. \end{aligned}$$

Solving $3c^2 - 12c + 8 = -3$ gives either $c = 2 + 2/\sqrt{3}$ or $c = 2 - 2/\sqrt{3}$ both of which lie in $[1,3]$.

We could have used the Mean Value Theorem to deduce that such a value of c exists since the function f satisfies the conditions of the Theorem on $[1,3]$.

ii) The formula (1) with $f(x) = 1/x^2$ on $[-1,1]$ becomes

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - 1}{2} = 0.$$

But $f'(c) = -2/c^3$ in which case $f'(c) = 0$ has no solution.

This time we could **not** apply the Mean Value Theorem to deduce that such a value of c exists. This is because the function f does **not** satisfy the conditions of the Theorem on $[-1,1]$. In particular $f(x)$ is not defined at $x = 0$ so it cannot be continuous there.

iii) The formula (1) with $f(x) = x^{1/3}$ on $[-1,1]$ becomes

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1.$$

But $f'(c) = c^{-2/3}/3$ and solving $c^{-2/3}/3 = 1$ gives $c = 1/3^{3/2} \approx 0.19245...$ which lies in $[-1,1]$.

Though this c exists we could **not** apply the Mean Value Theorem to deduce that such a value of c exists. Again the function f does **not** satisfy the conditions of the Theorem on $[-1,3]$. In particular $f(x)$ is not differentiable at $x = 0$ because

$$\lim_{x\to 0} \frac{x^{1/3}-0}{x-0} = \lim_{x\to 0} \frac{1}{x^{2/3}}$$

is not finite.

4. i) Show that

$$\left|\sin^2 b - \sin^2 a\right| \le |b-a|$$

for all $a, b \in \mathbb{R}$.

Careful. If you apply the Mean Value Theorem without thought you will get an unwanted factor of 2 on the right hand side.

ii) Show that

$$|\tan b - \tan a| \ge |b-a|$$

for all $a, b \in (-\pi/2, \pi/2)$.

iii) Show that

$$\cosh x \ge 1$$

for all $x \in \mathbb{R}$ and deduce

$$|\sinh b - \sinh a| \ge |b-a|$$

for all $a, b \in \mathbb{R}$.

iv) Show that

$$|\tanh b - \tanh a| \le |b-a|$$

for all $a, b \in \mathbb{R}$.

Solution i) Apply the Mean Value Theorem to $f(x) = \sin^2 x$ on $[a,b]$. But **note** that $f'(x) = 2\sin x\cos x = \sin 2x$ (this observation saves a factor of 2). So there exists $c \in (a,b)$ such that

$$\left|\frac{\sin^2 b - \sin^2 a}{b-a}\right| = |\sin 2c| \le 1.$$

ii) Start from

$$\frac{d\tan x}{dx}=\frac{1}{\cos^2 x},$$

which is well-defined for $x\in[a,b]\subseteq(-\pi/2,\pi/2)$. Thus by the Mean Value Theorem we have, for some $a<c<b$,

$$|\tan b-\tan a|=\frac{1}{\cos^2 c}\,|b-a|\geq|b-a|\,.$$

since $\cos^2 c\leq 1$.

iii) Since

$$\frac{d\sinh x}{dx}=\cosh x,$$

the Mean Value Theorem applied to $\sinh x$ on $[a,b]$ will give

$$|\sinh b-\sinh a|=|\cosh c|\,|b-a|$$

for some c between a and b. So the result will follow if $\cosh c\geq 1$.

The Hyperbolic cosine is given by

$$\cosh x=\frac{e^x+e^{-x}}{2}=\frac{e^x+\frac{1}{e^x}}{2}.$$

Then we have a sequence of reversible implications

$$\begin{aligned}\cosh x\geq 1\quad&\iff\quad\frac{e^x+\frac{1}{e^x}}{2}\geq 1\iff e^x+\frac{1}{e^x}\geq 2\\&\iff\quad(e^x)^2-2e^x+1\geq 0\\&\iff\quad(e^x-1)^2\geq 0.\end{aligned}$$

Reading backwards we see that the trivially true statement $(e^x-1)^2\geq 0$ implies the required $\cosh x\geq 1$.

iv) By the Mean Value Theorem applied to $\tanh x$ on the interval $[a,b]$ there exists $c\in(a,b)$:

$$|\tanh b-\tanh a|=\left|\frac{1}{\cosh^2 c}\right||b-a|\leq|b-a|\,,$$

the final inequality following from $\cosh c\geq 1$.

5. i) Prove, by using the Mean Value Theorem, that

   a) $\sin x$ is strictly increasing on $[-\pi/2, \pi/2]$,

   b) $\cos x$ is strictly decreasing on $[0, \pi]$,

   c) $\tan x$ is strictly increasing on $(-\pi/2, \pi/2)$.

   ii) What result from the notes do you need to quote to justify the existence of a function

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

   satisfying $\sin(\arcsin x) = x$ for all $x \in [-1, 1]$?

   iii) Explain how to define

   a) $\arccos : [-1, 1] \to [0, \pi]$,

   b) $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution i) a) Let $-\pi/2 \leq x < y \leq \pi/2$ be given. Then by the Mean Value Theorem applied to $\sin t$ on the interval $[x, y]$ there exists $c \in (x, y)$:

$$\sin y - \sin x = \cos c \times (y - x).$$

Since $\cos c > 0$ for $-\pi/2 < c < \pi/2$, i.e. *strictly* greater than 0, we have $\sin y - \sin x > 0$, i.e. $\sin y > \sin x$. Therefore sin is *strictly* increasing on $[-\pi/2, \pi/2]$.

b) Let $0 \leq x < y \leq \pi$ be given. Then by the Mean Value Theorem applied to $\cos t$ on the interval $[x, y]$ there exists $c \in (x, y)$:

$$\cos y - \cos x = -\sin c \times (y - x)$$

Since $\sin c > 0$ for $0 < c < \pi$, i.e. *strictly* greater than 0, we have $\cos y - \cos x < 0$, i.e. $\cos y > \cos x$. Therefore cos is *strictly* decreasing on $[0, \pi]$.

c) Let $-\pi/2 \leq x < y \leq \pi/2$ be given. Then by the Mean Value Theorem applied to $\tan t$ on the interval $[x, y]$ there exists $c \in (x, y)$:

$$\tan y - \tan x = \frac{1}{\cos^2 c} \times (y - x)$$

Since $\cos c \neq 0$ for $-\pi/2 < c < \pi/2$ we have $\tan y - \tan x > 0$, i.e. $\tan y > \tan x$. Therefore tan is *strictly* increasing on $(-\pi/2, \pi/2)$

ii) **To define** $\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$ all that is required is to verify that the conditions of the Inverse Function Theorem are satisfied.

First we need that sin is strictly monotonic on $[-\pi/2, \pi/2]$. By Part 1.a. it is in fact strictly increasing. Given it is increasing we need then check that $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ which are, in fact, true. Hence the Inverse Function Theorem can be applied to $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ to give the existence of $\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$.

iii) a. **To define** $\arccos : [-1, 1] \to [0, \pi]$ all that is required is to verify that the conditions of the Inverse Function Theorem are satisfied.

First we need that cos is strictly monotonic on $[0, \pi]$. By Part 1.b. it is in fact strictly decreasing. Given it is decreasing we need then check that $\cos 0 = 1$ and $\cos(\pi) = -1$ which are, in fact, true. Hence the Inverse Function Theorem can be applied to $\cos : [0, \pi] \to [-1, 1]$ to give the existence of $\arccos : [-1, 1] \to [0, \pi]$.

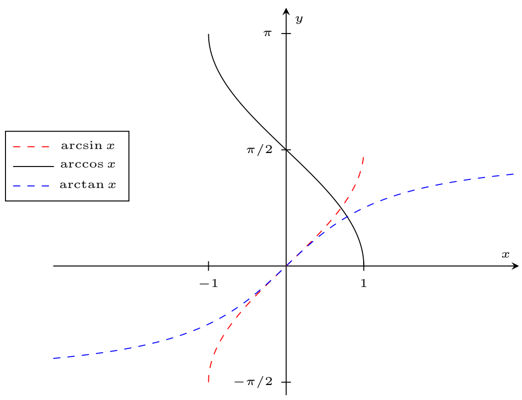
b. **To define** $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ you require an extension of the Inverse Function Theorem (as stated but not proved in the notes) to unbounded sets.

This requires $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ to be strictly monotonic. By part i.c. tan is, in fact, strictly increasing. Given any $\delta > 0$ the Inverse Function Theorem can be applied to tan on the closed interval $[-\pi/2 + \delta, \pi/2 - \delta]$ to give the function

$$\arctan : [\tan(-\pi/2 + \delta), \tan(\pi/2 - \delta)] \to [-\pi/2 + \delta, \pi/2 - \delta].$$

Let $\delta \to 0$ when $\tan(-\pi/2 + \delta) \to -\infty$ and $\tan(\pi/2 - \delta) \to +\infty$ and we have the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$.

Perhaps in the following figure you can see that arcsin and arctan are strictly increasing while arctan is strictly decreasing.



You should also be able to see that the inverse trig functions map between the sets claimed above. So $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$, $\arccos : [-1,1] \to [0,\pi]$ and $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$.

6. Using the Mean Value Theorem prove that

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \tag{2}$$

for $x > 0$.

(If necessary you may assume a result from the lectures that $e^x > 1+x$ for all $x \neq 0$.)

Hint Define an appropriate functions of t on the interval $[0,x]$, one for the lower bound in (2) the other for the upper.

Solution For the lower bound on $\ln(1+x)$ define

$$g(t) = \ln(1+t) - t + \frac{t^2}{2}$$

for $t > 0$. Then

$$g'(t) = \frac{1}{1+t} - 1 + t = \frac{1-(1-t^2)}{1+t} = \frac{t^2}{1+t} > 0 \tag{3}$$

for $t > 0$. Given any $x > 0$, apply the Mean Value Theorem to g on the interval $[0,x]$, which gives the existence of some $c \in (0,x)$ for which

$g(x) - g(0) = g'(c)\,x$. Yet $g'(c) > 0$ by (3), which with $x > 0$ implies $g(x) > g(0) = 0$ which leads to the required result.

For the upper bound on $\ln(1+x)$ let

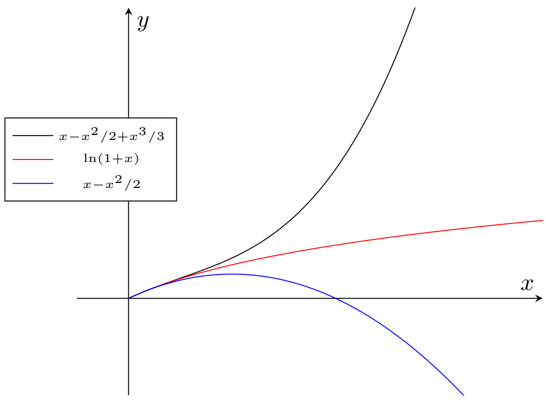
$$f(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ln(1+t)$$

for $t > 0$. Then

$$\begin{aligned} f'(t) &= 1 - t + t^2 - \frac{1}{1+t} = \frac{(1+t)(1-t+t^2)-1}{1+t} \\ &= \frac{t^3}{1+t} > 0 \qquad (4) \end{aligned}$$

for $t > 0$. Given any $x > 0$, apply the Mean Value Theorem to f on the interval $[0, x]$, which gives the existence of some $c \in (0, x)$ for which $f(x) - f(0) = f'(c)\,x$. Yet $f'(c) > 0$ by (4), which with $x > 0$ implies $f(x) > f(0) = 0$ and this leads to the required result.

Figure for Question 6,



7. State the Cauchy Mean Value Theorem.

Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that

i) there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \left(e^{b-c} - e^{a-c}\right) f'(c).$$

Hint Rearrange this equality so that a and b occur on one side, c on the other. You should then deduce what function has to be chosen for g in Cauchy's Mean Value.

ii) Assuming further that $f(x) > 0$ and $f'(x) \neq 0$ on $[a, b]$ prove that there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \ln\left(\frac{f(b)}{f(a)}\right) f(c).$$

Strangely no derivatives appear anywhere!

iii) Assuming that $f'(x) \neq 0$ for all $x \in (a, b)$ prove that there exists $c \in (a, b)$ such that

$$f(b) = f(a) + e^{f(b)-f(c)} - e^{f(a)-f(c)}.$$

Again no derivatives appear.

Solution ***Cauchy's Mean Value Theorem***: *Assume that f and g are continuous on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

i) Rearrange the given equality as

$$\frac{f(b) - f(a)}{e^b - e^c} = \frac{f'(c)}{e^c}.$$

In the Cauchy Mean Value Theorem the right hand side is $f'(c)/g'(c)$ so we need $g'(c) = e^c$. To ensure this holds at the unknown point c we demand that $g'(x) = e^x$ for **all** $x \in (a, b)$. Integrate to get $g(x) = e^x$ (choose the constant of integration to be 0). I leave it to the student to check that this works.

ii) *Rough work* Rearrange the given equality as

$$\frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} = f(c)$$

In the Cauchy Mean Value Theorem the right hand side is $f'(c)/g'(c)$ so we need $f'(c)/g'(c) = f(c)$. To ensure this holds at the unknown point c we demand that

$$\frac{f'(x)}{g'(x)} = f(x), \quad \text{i.e.} \quad g'(x) = \frac{f'(x)}{f(x)}$$

for all $x \in (a, b)$. Integrate to get $g(x) = \ln f(x)$. *End of Rough work.*

Proof Apply the Cauchy Mean Value Theorem with $g(x) = \ln f(x)$, allowable since $f(x) > 0$ on $[a, b]$. Then for some c between a and b,

$$\frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} = \frac{f'(c)}{(f'(c)/f(c))} = f(c),$$

using

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}.$$

Finally note that $\ln f(b) - \ln f(a) = \ln(f(b)/f(a))$ to get required result on rearrangement.

iii. This time choose $g(x) = e^{f(x)}$. Note that $g'(x) = f'(x)\, e^{f(x)} \neq 0$ for all x since we are told that $f'(x) \neq 0$ for all x. So we can apply the Cauchy Mean Value Theorem to find $c \in (a, b)$ for which

$$\frac{f(b) - f(a)}{e^{f(b)} - e^{f(a)}} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} = \frac{f'(c)}{f'(c)\, e^{f(c)}} = e^{-f(c)}.$$

This rearranges to the given result.

Inverses

We have proven the existence of the inverse of trig functions in Question 5 and of the hyperbolic trig functions in Question 7, Sheet 5. We now ask if they are differentiable.

8. State the Theorem on the Differentiation of Inverse functions.

   i) Prove that
   $$\cos(\arcsin y) = \sqrt{1-y^2}$$
   for $-1 < y < 1$.

   Hint: Try to use $y = \sin(\arcsin y)$ by relating cos to sin through $\sin^2\theta + \cos^2\theta = 1$ for all θ.

   ii) Use the Theorem on the Differentiation of Inverse functions to prove that
   $$\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}}$$
   on $(-1, 1)$,

   iii) Give similar results for

   a) $\dfrac{d}{dy}\arccos y \quad \text{on } (-1,1)$,

   b) $\dfrac{d}{dy}\arctan y \quad \text{on } \mathbb{R}$.

Solution ***Inverse Rule*** *Suppose that f is strictly monotonic and continuous on a closed and bounded interval $[a,b]$. Write*

$$[c,d] = \begin{cases} [f(a), f(b)] & \text{if } f \text{ is increasing} \\ [f(b), f(a)] & \text{if } f \text{ is decreasing.} \end{cases}$$

By the Inverse Function Theorem there exists a strictly monotonic, continuous function $g : [c,d] \to [a,b]$ which is the inverse function of f so, if $y = f(x)$ then $x = g(y)$.

Suppose that f is differentiable at $\ell \in (a,b)$ with $df/dx \neq 0$ at $x = \ell$. Write $k = f(\ell)$ so $\ell = g(k)$ and $k \in (c,d)$.

Then g is differentiable at k and

$$\left.\frac{dg(y)}{dy}\right|_{y=k} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=\ell}}, \quad \text{i.e.} \quad \frac{dg}{dy}(k) = \frac{1}{\frac{df}{dx}(\ell)} = \frac{1}{\frac{df}{dx}(g(k))}.$$

If f is differentiable on (a,b) with $df/dx \neq 0$ at all points of (a,b), then g is differentiable on (c,d) and

$$\frac{dg(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=g(y)}} = \frac{1}{\frac{df}{dx}(g(y))}, \tag{5}$$

for all $y \in (c,d)$.

i) Starting with $y = \sin(\arcsin y)$ we have

$$1 - y^2 = 1 - \sin^2(\arcsin y) = \cos^2(\arcsin y).$$

We take square roots but we need worry about signs. From the definition in Question 5 we have$-\pi/2 \leq \arcsin y \leq \pi/2$ and for such angles the cosine is positive. Thus we take the positive root to get

$$\cos(\arcsin y) = \sqrt{1-y^2}.$$

ii) The function $g : [-1,1] \to [-\pi/2, \pi/2]$, $g(y) = \arcsin y$, is the inverse function of $f(x) = \sin x$. From the Theorem on the Differentiation of Inverse functions, and (5) in particular, we deduce that $g(y)$ is differentiable on $(-1,1)$ with

$$\begin{aligned}\frac{d}{dy}\arcsin y &= \frac{dg(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=g(y)}} = \frac{1}{\left.\frac{d\sin x}{dx}\right|_{x=\arcsin y}} \\ &= \frac{1}{\cos(\arcsin y)} = \frac{1}{\sqrt{1-y^2}},\end{aligned}$$

by part i., for $y \in (-1,1)$.

iii) a. Similarly, the function $g : [-1,1] \to [0,\pi]$, $g(y) = \arccos y$, is the inverse function of $f(x) = \cos x$. From the Theorem on the

Differentiation of Inverse functions we deduce that g is differentiable on $(-1,1)$ with

$$\begin{aligned}\frac{d}{dy}\arccos y &= \frac{dg(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=g(y)}} = \frac{1}{\left.\frac{d\cos x}{dx}\right|_{x=\arccos y}} \\ &= -\frac{1}{\sin(\arccos y)}.\end{aligned}$$

for $y \in (-1,1)$. As in part i. it can be shown that $\sin(\arccos y) = \sqrt{1-y^2}$ hence

$$\frac{d}{dy}\arccos y = -\frac{1}{\sqrt{1-y^2}}$$

for $y \in (-1,1)$.

iii) b. The function $g : \mathbb{R} \to (-\pi/2, \pi/2)$, $g(y) = \arctan y$, is the inverse function of $f(x) = \tan x$. From the Theorem on the Differentiation of Inverse functions we deduce that g is differentiable on $\mathbb{R}$ with

$$\frac{d}{dy}\arctan y = \cos^2(\arctan y)$$

for $y \in \mathbb{R}$. To simplify the right hand side, let $w = \arctan y$ so

$$y = \tan w = \frac{\sin w}{\cos w} = \frac{\sqrt{1-\cos^2 w}}{\cos w}.$$

Rearrange to get $\cos^2 w = 1/(1+y^2)$. Hence

$$\frac{d}{dy}\arctan y = \frac{1}{1+y^2}$$

for $y \in \mathbb{R}$.

9. Prove that

$$\arctan x > \frac{x}{1+x^2/3}$$

for $x > 0$.

Hint This question is here as an application of the derivative of arctan found in the previous question.

Solution Define

$$g(t) = \arctan t - \frac{t}{1 + t^2/3}.$$

for $t > 0$. Then, by the previous question,

$$\begin{aligned} g'(t) &= \frac{1}{1+t^2} - \frac{(1 + t^2/3) - 2t^2/3}{(1+t^2/3)^2} \\ &= \frac{(1+t^2/3)^2 - (1+t^2)(1-t^2/3)}{(1+t^2)(1+t^2/3)^2} \\ &= \frac{4t^4}{(1+t^2)(3+t^2)^2} > 0. \end{aligned}$$

Given $x > 0$ apply the Mean Value Theorem to g on $[0, x]$ to find $c \in (0, x)$ such that

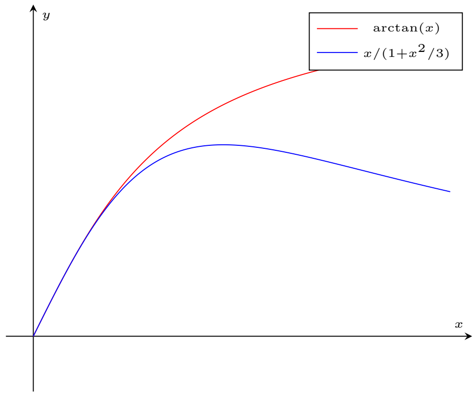
$$g(x) - g(0) = \frac{4c^4}{9(1+c^2)(1+c^2/3)^2}(x - 0) > 0.$$

Thus $g(x) > g(0) = 0$, i.e.

$$\arctan x - \frac{x}{1 + x^2/3} > 0$$

for $x > 0$.

Figure for Question 9,



10. i) a) Prove that

$$\cosh\left(\sinh^{-1} y\right) = \sqrt{1+y^2}$$

for all $y \in \mathbb{R}$.

b) Use the Theorem on the Differentiation of Inverse functions to prove that

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{\sqrt{1+y^2}}$$

for all $y \in \mathbb{R}$.

ii) Prove that

$$\frac{d}{dy}\cosh^{-1} y = \frac{1}{\sqrt{y^2-1}}$$

for $y > 1$.

iii) Evaluate

$$\frac{d}{dy}\tanh^{-1} y$$

for $y \in (-1, 1)$.

Solution i) a) From the known identity $\cosh^2 x - \sinh^2 x = 1$ (and if you don't know it, prove it) with $x = \sinh^{-1} y$ we get

$$\cosh^2\left(\sinh^{-1} y\right) = 1 + \sinh^2\left(\sinh^{-1} y\right) = 1 + y^2.$$

As seen earlier, $\cosh x \geq 1$, and in particular it is positive so we take the positive square root to get

$$\cosh\left(\sinh^{-1} y\right) = \sqrt{1+y^2},$$

for all $y \in \mathbb{R}$.

b) The function $g : \mathbb{R} \to \mathbb{R}$, $g(y) = \sinh^{-1} y$, is the inverse of $f(x) = \sinh x$. From the Inverse Function Theorem we deduce that g is differentiable on $\mathbb{R}$ with

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{\left.\frac{d\sinh x}{dx}\right|_{x=\sinh^{-1} y}} = \frac{1}{\cosh\left(\sinh^{-1} y\right)} = \frac{1}{\sqrt{1+y^2}},$$

by Part i.a., for all $y \in \mathbb{R}$.

ii) The function $g : [1,\infty) \to [0,\infty)$, $g(y) = \cosh^{-1} y$, is the inverse of $f(x) = \cosh x$. From the Inverse Function Theorem we deduce that g is differentiable on $[1,\infty)$ with

$$\frac{d}{dy}\cosh^{-1} y = \frac{1}{\left.\frac{d\cosh x}{dx}\right|_{x=\cosh^{-1} y}} = \frac{1}{\sinh\left(\cosh^{-1} y\right)}$$

for $y > 1$. To simplify the answer note that, as above,

$$\sinh^2\left(\cosh^{-1} y\right) = \cosh^2\left(\cosh^{-1} y\right) - 1 = y^2 - 1.$$

Since $\cosh^{-1} y$ has a non-negative image for any y, then $\sinh\left(\cosh^{-1} y\right)$ will be positive and so we need take a positive square root to get

$$\frac{d}{dy}\cosh^{-1} y = \frac{1}{\sqrt{y^2-1}},$$

for $y > 1$.

iii) The function $g : (-1,1) \to \mathbb{R}$, $g(y) = \tanh^{-1} y$, is the inverse of $f(x) = \tanh x$. From the Inverse Function Theorem we deduce that g is differentiable on $(-1,1)$ with

$$\begin{aligned}\frac{d}{dy}\tanh^{-1} y &= \frac{1}{\left.\frac{d\tanh x}{dx}\right|_{x=\tanh^{-1} y}} = \frac{1}{\left(1/\cosh^2\left(\tanh^{-1} y\right)\right)} \\ &= \cosh^2\left(\tanh^{-1} y\right).\end{aligned}$$

To simplify the answer, start from $\cosh^2 x - \sinh^2 x = 1$ and take out a factor of $\cosh^2 x$ so

$$\cosh^2 x \left(1 - \tanh^2 x\right) = 1 \text{ and thus } \cosh^2 x = \frac{1}{1 - \tanh^2 x}.$$

Then substituting $x = \tanh^{-1} y$ gives

$$\cosh^2 \left(\tanh^{-1} y\right) = \frac{1}{1 - \tanh^2 \left(\tanh^{-1} y\right)} = \frac{1}{1 - y^2}.$$

Hence

$$\frac{d}{dy} \tanh^{-1} y = \frac{1}{1 - y^2}$$

for $y \in (-1, 1)$.

L'Hôpital's and Chain Rule

11. State L'Hôpital's Rule.

    Use L'Hôpital's Rule to evaluate

    i)
    $$\lim_{x \to 0} \frac{\ln(1 - x) + \ln(1 + x)}{x^2},$$

    ii)
    $$\lim_{x \to 0} \frac{(1 + x)\ln(1 - x) - (1 - x)\ln(1 + x) + 2x}{x^3},$$

    iii)
    $$\lim_{x \to 0} \frac{\arcsin x - x}{x^3},$$

    iv)
    $$\lim_{x \to 0} \frac{\sinh^{-1} x - x}{x^3}.$$

Solution ***L'Hôpital's Rule*** *Suppose that f and g are both differentiable and $g'(x) \neq 0$ on some deleted neighbourhood of a. Then if $f(a) = g(a) = 0$ and $\lim_{x\to a} f'(x)/g'(x)$ exists, we have*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

i) When $x = 0$ both numerator and denominator are zero, so we can apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{\ln(1-x) + \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{-\frac{1}{1-x} + \frac{1}{1+x}}{2x}.$$

Though when $x = 0$ both numerator and denominator are again zero, do not use L'Hôpital's Rule but rather simplify

$$\frac{-\frac{1}{1-x} - \frac{1}{1+x}}{2x} = -\frac{2x}{2x(1-x)(1+x)} = -\frac{1}{1-x^2} \to -1.$$

as $x \to 0$.

ii) When $x = 0$ both numerator and denominator are zero, so we can apply L'Hôpital's Rule to get

$$\lim_{x\to 0} \frac{(1+x)\ln(1-x) - (1-x)\ln(1+x) + 2x}{x^3}$$
$$= \lim_{x\to 0} \frac{\ln(1-x) - \frac{1+x}{1-x} + \ln(1+x) - \frac{1-x}{1+x} + 2}{3x^2}. \qquad (6)$$

Combining

$$-\frac{1+x}{1-x} - \frac{1-x}{1+x} + 2 = -\frac{4x^2}{(1-x^2)},$$

and using the sum rule for limits, the limit (6) equals

$$\frac{1}{3}\lim_{x\to 0} \frac{\ln(1-x) + \ln(1+x)}{x^2} - \frac{4}{3}\lim_{x\to 0} \frac{1}{1-x^2} = -\frac{1}{3} - \frac{4}{3} = -\frac{5}{3},$$

having used Part i.

iii) When $x = 0$ both numerator and denominator are zero, so we can apply L'Hôpital's Rule. Using Question 8 we have

$$\lim_{x\to 0} \frac{\arcsin x - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2} = \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}}.$$

Do not use L'Hôpital's again but note that

$$1-\sqrt{1-x^2} = 1-\sqrt{1-x^2} \times \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \frac{x^2}{1+\sqrt{1-x^2}}. \qquad (7)$$

Hence

$$\lim_{x\to 0} \frac{\arcsin x - x}{x^3} = \lim_{x\to 0} \frac{1}{3\left(1+\sqrt{1-x^2}\right)\sqrt{1-x^2}} = \frac{1}{6}.$$

iv) When $x = 0$ both numerator and denominator are zero, so we can apply L'Hôpital's Rule. Using Question 10 we find that

$$\begin{aligned}\lim_{x\to 0} \frac{\sinh^{-1} x - x}{x^3} &= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+x^2}} - 1}{3x^2} = \lim_{x\to 0} \frac{1-\sqrt{1+x^2}}{3x^2\sqrt{1+x^2}} \\ &= -\lim_{x\to 0} \frac{1}{3\left(1+\sqrt{1+x^2}\right)\sqrt{1+x^2}},\end{aligned}$$

using an appropriate version of (7). Hence

$$\lim_{x\to 0} \frac{\sinh^{-1} x - x}{x^3} = -\frac{1}{6}.$$

12. State the Chain Rule for Differentiation.

Prove

i)

$$\frac{d}{dy}\left(\tanh^{-1}(\sin y)\right) = \frac{1}{\cos y}$$

for $y \in (-\pi/2, \pi/2)$.

ii)

$$\frac{d}{dy}\left(\sinh^{-1}(\tan y)\right) = \frac{1}{\cos y}$$

for $y \in (-\pi/2, \pi/2)$.

iii) Can you give a similar example for $\cosh^{-1}$ with an appropriate trigonometric function?

Solution ***Chain or Composite Rule*** *If g is differentiable at a and f is differentiable at $g(a)$ then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))\, g'(a).$$

i) Recall from Question 10 that

$$\frac{d}{dy}\left(\tanh^{-1} y\right) = \frac{1}{1-y^2}$$

for $-1 < y < 1$. By the Composite Rule,

$$\frac{d}{dy}\left(\tanh^{-1}(\sin y)\right) = \frac{1}{1-(\sin y)^2}\cos y = \frac{\cos y}{\cos^2 y} = \frac{1}{\cos y}.$$

This is valid as long as $|\sin y| < 1$, (i.e. $|\sin y| \neq 1$) which is certainly true for $y \in (-\pi/2, \pi/2)$.

ii) Recall from Question 10 that

$$\frac{d}{dy}\left(\sinh^{-1} y\right) = \frac{1}{\sqrt{1+y^2}}$$

for $y \in \mathbb{R}$. By the Composite Rule,

$$\frac{d}{dy}\left(\sinh^{-1}(\tan y)\right) = \frac{1}{\sqrt{1+(\tan y)^2}} \times \frac{1}{\cos^2 y} = \frac{\cos y}{\cos^2 y} = \frac{1}{\cos y}.$$

True as long as $\cos y \neq 0$ and so certainly true for $y \in (-\pi/2, \pi/2)$.

iii) Recall from Question 10 that

$$\frac{d}{dy}\cosh^{-1} y = \frac{1}{\sqrt{y^2-1}}$$

for $y > 1$. Then, by the Composition Rule,

$$\frac{d}{dy}\cosh^{-1}\left(\frac{1}{\cos y}\right) = \frac{1}{\sqrt{\left(\frac{1}{\cos y}\right)^2 - 1}} \times -\frac{-\sin y}{\cos^2 y} = \frac{1}{\cos y}.$$

But be careful. For the first equality to hold we need note that $1/\cos y > 1$, i.e. $0 < \cos y < 1$. In using $\sqrt{1-\cos^2 y} = \sin y$ we are taking the positive root, i.e. $\sin y > 0$. The restrictions $0 < \cos y < 1$ and $\sin y > 0$ are simultaneously satisfied if $y \in (0, \pi/2)$.

Additional Questions

13. Using the Mean Value Theorem prove that

$$\frac{1}{1 - x + \dfrac{x^2}{3}} > e^x > \frac{1}{1 - x + \dfrac{x^2}{2}},$$

the first inequality for $0 < x < 1$, the second for all $x > 0$.

Hint Examine each inequality sparately and multiply up.

Solution On multiplying up it suffices to prove

$$1 > e^x\left(1 - x + \frac{x^2}{3}\right)$$

for $0 < x \le 1$ and

$$e^x\left(1 - x + \frac{x^2}{2}\right) > 1$$

for $x > 0$.

i. Define

$$f(t) = 1 - e^t\left(1 - t + \frac{t^2}{3}\right).$$

It suffices to show that $f(x) > 0$ for $0 < x \leq 1$. Given $x : 0 < x \leq 1$ apply the Mean Value Theorem to f on the interval $[0, x]$ to find c : $0 < c < x \leq 1$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) .$$

But

$$f'(t) = -e^t \left(1 - t + \frac{t^2}{3}\right) - e^t \left(-1 + \frac{2}{3}t\right) = \frac{1}{3}e^t t\,(1 - t) .$$

Hence

$$f(x) - f(0) = f'(c)\,x = \frac{1}{3}e^c c\,(1 - c)\,x > 0,$$

since $x > 0$ and $1 > c > 0$. Yet $f(0) = 0$ so the result follows.

ii. Let

$$g(t) = \left(1 - t + \frac{t^2}{2}\right) e^t - 1.$$

It suffices to show that $g(x) > 0$ for $x > 0$. Given $x > 0$ apply the Mean Value Theorem to g on the interval $[0, x]$ to find $c : 0 < c < x \leq 1$ such that

$$g(x) - g(0) = g'(c)\,(x - 0) = \frac{1}{2}e^c c^2 x > 0$$

since $c > 0$. Again $g(0) = 0$ and the result follows.

14. An example of the use of L'Hôpital's Rule, stated but left to the student to check, was

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Use this result to show that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$ and find the value of the derivative at $x = 0$.

Solution Go back to the definition and for $x \neq 0$ consider

$$\frac{f(x)-f(0)}{x-0} = \frac{\frac{\sin x}{x}-1}{x} = \frac{\sin x - x}{x^2} = x\frac{\sin x - x}{x^3}.$$

Thus, by the Product Rule,

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0} x\frac{\sin x - x}{x^3} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\frac{\sin x - x}{x^3}\right)\\ &= 0\times\left(-\frac{1}{6}\right) = 0.\end{aligned}$$

Since the limit exists the function f is differentiable at $x = 0$ with derivative 0.